A remark on credit risk models and copula

KUSUOKA Shigeo (Univ. Tokyo)

and

Takenobu NAKASHIMA(Nomura Sec. Co. , Univ. Tokyo)

Copopula Models : describe joint distribution of stopping times

(default times)

$$\Rightarrow$$

Risk Management : Static

Pricing Credit Derivatives : Dynamic

Typical Coupula Models: Gaussian, Gumbel, inverse Gunbel etc.

Question

Are these copula models dynamically consistent ?

Answer: Probably not

Björk-Christernsen(1999) Math. Fin. 9, 323-348

"Interestrate Dynamics and Consistent Forwardrate Curves"

Family of Forward Rate Curves

Dynamical Interest Rate model free of arbitrage

$\exists$ some analytic constraint conditions

Mathematical Setting

$(\Omega, \mathcal{F}, P)$: complete probability space

$W(t) = (W^1(t), \dots, W^d(t))$, $t \geqq 0$, d-dim. Wiener process

$\mathcal{G}_t = \sigma\{W(s),\ s \in [0,t]\} \vee \mathcal{N}$

$\mathcal{N} = \{B \in \mathcal{F};\ P(B) = 0 \text{ or } 1\}$

$N \geqq 2$

$\tau_i : \Omega \to [0, \infty)$, $i = 1, \dots, N$, random variables

$\mathcal{F}_t = \mathcal{G}_t \vee \sigma\{\tau_i \wedge t, i = 1, \dots, N\}$

Basic Assumptions

$\xi_i : [0,\infty) \times \Omega \to [0,\infty)$, $i = 1,\ldots,N$, $\mathcal{G}$-prog. m'ble

(SC) $$(\prod_{i \in I} 1_{\{\tau_i > t\}}) P(\tau_i > t_i, i \in I | \mathcal{F}_t)$$

$$= (\prod_{i \in I} 1_{\{\tau_i > t\}}) E[\exp(-\sum_{i \in I} \int_t^{t_i} \xi_i(s) ds) | \mathcal{G}_t] \ a.s.$$

for any $I \subset \{1,\ldots,N\}$ and $t, t_i \in [0,\infty), i \in I$ with $t \leqq \min_{i \in I} t_i$

(PO) For any $t \geqq 0$,

$$P(\bigcap_{i=1}^{N} \{\tau_i > t\}) | \mathcal{G}_t) > 0 \ a.s.$$

Technical Assumptions

(A-1) For any $T > 0$,

$$\sum_{i=1}^{N} \int_0^T E[\xi_i(t)^4]dt < \infty.$$

(A-2) For any $i = 1, \ldots, N$,

$$\int_0^\infty \xi_i(t) = \infty \quad a.s. \text{ and } \int_a^b \xi_i(t) > 0 \quad a.s. \text{ for any } a, b > 0 \text{ with } b > a.$$

(A-3) $$\sum_{i=1}^{N} \int_0^\infty (1+t)^2 E[\xi_i(t)^2 \exp(-2 \int_0^t \xi_i(s)ds)]dt < \infty.$$

$\theta : [0,\infty) \times \Omega \to \mathbf{R}^M$ $\mathcal{G}$-Ito process

i.e.,

θ : $\mathcal{G}$-progressively measurable

$\theta(t,\omega)$ continuous in t for all $\omega \in \Omega$

$\exists \eta_k : [0,\infty) \times \Omega \to \mathbf{R}^M$ $k = 0, 1, \ldots, d$, $\mathcal{G}$-prog. m'ble

$$P\left(\sum_{k=1}^{d} \int_0^T |\eta_k(t)|^2 dt + \int_0^T |\eta_0(t)| dt < \infty\right) = 1 \text{ for any } T > 0,$$

$$\theta(t) = \theta(0) + \sum_{k=1}^{d} \int_0^t \eta_k(s) dW^k(s) + \int_0^t \eta_0(s) ds$$

Θ open subset in $\mathbf{R}^M$

(A-4) $P(\theta(t) \in \bar{\Theta}$ for all $t \geqq 0) = 1$

(A-5) the support of probability law of $\theta(t,\omega)$ under $e^{-t}dt \otimes P(d\omega)$ contains a non-empty open set in Θ

i.e., $\exists\, U_0$ non-empty open set $\quad U_0 \subset \Theta$

$$\forall \theta_0 \in U_0 \ \forall \varepsilon > 0 \quad \int_0^\infty P(|\theta(t) - \theta_0| < \varepsilon)\, e^{-t} dt > 0$$

Copula Assumption

$\exists K \in C([0,1]^N \times \Theta; [0,1])$

(CP) $K(\cdot,\theta) : [0,1]^N \to [0,1]$ is a copula function for any $\theta \in \Theta$, and

$$(\prod_{i=1}^{N} 1_{\{\tau_i > t\}}) 1_\Theta(\theta(t)) P(\tau_i > t_i, i = 1, \ldots, N | \mathcal{F}_t)$$

$$= (\prod_{i=1}^{N} 1_{\{\tau_i > t\}}) 1_\Theta(\theta(t)) K(P(\tau_1 > t_1 | \mathcal{F}_t), \ldots, P(\tau_N > t_N | \mathcal{F}_t), \theta(t)) \ a.s.$$

for any $t, t_1, \ldots, t_N > 0$ with $t < \min_{i=1,\ldots,N} t_i$.

$$((\Omega, \mathcal{F}, P), (W_t^k)_{k=1,\dots,d}, (\tau_i)_{i=1,\dots,N}, (\xi_i(t))_{i=1,\dots,N}, \theta(t), \Theta, K)$$

is called a dynamical default time copula model

and we call

K the associated family of copula functions to this model.

Definition 1 Let Θ be an open subset in $\mathbf{R}^M$. We say that $K \in C([0,1]^N \times \Theta; [0,1])$ is an admissible family of copula functions, if there is a dynamical default time copula model and K is the associated family of copula functions to the model.

$N \geqq 2,\ M \geqq 1$

Θ a non-void open subset in $\mathbf{R}^M$

$\mathcal{C}_{(N)}(\Theta)$:

the set of $K \in C([0,1]^N \times \Theta; [0,1])$ such that

$K(\cdot, \theta) : [0,1]^N \to [0,1]$ is a copula function for any $\theta \in \Theta$,

and $K|_{(0,1)^N \times \Theta}$ is a C^∞ function.

metric dis of $\mathcal{C}_{(N)}(\Theta)$

D_n, $n \geqq 1$, an increasing sequence of compact subsets in Θ with $\bigcup_{n=1}^{\infty} D_n = \Theta$

$$dis(K_1, K_2)$$

$$= \sum_{n=1}^{\infty} 2^{-n} \wedge \sup\{|K_1(x,\theta) - K_2(x,\theta)|;\ x \in [0,1]^N,\ \theta \in D_n\}$$

$$+ \sum_{n=1}^{\infty} 2^{-n} \wedge \Big(\sum_{\alpha_1, \dots, \alpha_{N+M}=0}^{n} \sup\{| \frac{\partial^{\alpha_1 + \cdots \alpha_{N+M}} (K_1 - K_2)}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N} \partial \theta_1^{\alpha_{N+1}} \partial \dots \theta_M^{\alpha_{N+M}}} (x,\theta)|;$$

$$x \in [1/4n, 1 - 1/4n]^N,\ \theta \in D_n\}\Big)$$

$(\mathcal{C}_{(N)}(\Theta), dis)$ Polish space

Theorem 2 Let $N \geqq 3$, $M \geqq 1$, and Θ be a non-void open subset in $\mathbf{R}^M$. Then the subset of $\mathcal{C}_{(N)}(\Theta)$ whose elements are admissible families of copula functions is a set of the first category in Baire's sense.

a family of Gumbel copula functions

$N = 3,\ M = 1,\ \Theta = (0,1).$

$$K_G(x_1, x_2, x_3, \theta) = \exp(-(\sum_{i=1}^{3}(-\log x_i)^{\theta})^{1/\theta}), \qquad x_1, x_2, x_3 \in (0,1),\ \theta \in (0,1).$$

a family of inverse Gumbel copula functions

$N = 3$, $M = 1$, $\Theta = (0, 1)$.

$$K_{IG}(x_1, x_2, x_3, \theta)$$

$$= 1 - \sum_{i=1}^{3}(1 - x_i) + K_G(1, 1 - x_2, 1 - x_3) + K_G(1 - x_1, 1, 1 - x_3)$$

$$+K_G(1 - x_1, 1 - x_2, 1) - K_G(1 - x_1, 1 - x_2, 1 - x_3)$$

By numerical computation, we see that

K_G or K_{IG} are not admissible.

Main Idea

$\exists f_i : [0,\infty) \times [0,\infty) \times \Omega \to (0,\infty)$, $i = 1, \ldots, N$, such that

$$f_i(t,s) = E[\exp(-\int_{t\wedge s}^{s} \xi_i(r)dr)|\mathcal{G}_t] \quad a.s. \qquad t, s \geqq 0,$$

$\forall \omega \in \Omega$

$f_i(\cdot, *, \omega) : [0,\infty) \times [0,\infty)$ is continuous

$f_i(t, s_1, \omega) > f_i(t, s_2, \omega) > 0 \ s_2 > s_1 > t$

$$f(t,t,\omega) = 1, \quad \lim_{s\uparrow\infty} f(t,s,\omega) = 0, \qquad t \geqq 0, \ \omega \in \Omega.$$

$\exists \tilde{\sigma}_{i,k} : [0,\infty] \times [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1,\dots,d$, $i = 1,\dots,N$, such that

(1) $\tilde{\sigma}_{i,k}(t,\cdot,\omega) : [0,\infty] \to \mathbf{R}$, $k = 1,\dots,d$, is continuous

for any $t \in [0,\infty)$ and $\omega \in \Omega$.

(2) $\tilde{\sigma}_{i,k}(t,s,\omega) = 0$, $t \geqq s$, and $\lim_{s\to\infty} \tilde{\sigma}_{i,k}(t,s,\omega) = 0$

for any $t \in [0,\infty)$ and $\omega \in \Omega$.

(3) $\sigma_{i,k}(\cdot,s) : [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1,\dots,N$, is $\mathcal{G}$-prog. m'ble

(4) For any $s > 0$

$$f_i(t,s) = f_i(0,s) + \int_0^{t\wedge s} \xi_i(r) f_i(r,s)dr + \sum_{k=1}^{d} \int_0^t \tilde{\sigma}_{i,k}(r,s)dW^k(r).$$

$T_i : [0, \infty) \times (0, 1) \times \Omega \to (0, \infty)$, $i = 1, \ldots, N$,

$$T_i(t, x) = \inf\{s \geqq t, f_i(t, s) \leqq x\}, \qquad x \in [0, 1).$$

$T_i(t, \cdot, \omega) : (0, 1) \to (0, \infty)$ is continuous and strictly decreasing, $\lim_{x \downarrow 0} T_i(t, x, \omega) = \infty$ and $\lim_{x \uparrow 1} T_i(t, x, \omega) = 0$ for any $t \geqq 0$ and $\omega \in \Omega$.

$\sigma_{i,k} : [0,\infty) \times (0,1) \times \Omega \to \mathbf{R},\ i = 1, \dots, N,\ k = 1, \dots, d,$

$$\sigma_{i,k}(t,x) = \tilde{\sigma}_{i,k}(t, T_i(t,x)) \qquad t \geqq 0,\ x \in (0,1).$$

$$\lim_{x \downarrow 0} \sigma_{i,k}(t,x) = 0, \qquad \lim_{x \uparrow 1} \sigma_{i,k}(t,x) = 0.$$

Extended to $\sigma_{i,k} : [0,\infty) \times [0,1] \times \Omega \to \mathbf{R}$

for which $\sigma_{i,k}(t,\cdot,\omega) : [0,1] \to \mathbf{R}$ is continuous for any $t \geqq 0$, $\omega \in \Omega$,

and $\sigma_{i,k}(t,0) = \sigma_{i,k}(t,1) = 0$.

Rough Idea

$$f_i(t,s) = P(\tau_i > s|\mathcal{F}_t), \qquad s > t,\ i = 1,\ldots,N,$$

$$E[\exp(-\sum_{i=1}^{N}\int_0^{s_i}\xi_i(r)dr)|\mathcal{G}_t]$$

$$= \exp(-\sum_{i=1}^{N}\int_0^{t}\xi_i(r)dr)P(\tau_1 > s_1,\ldots,\tau_N > s_n|\mathcal{F}_t)$$

$$= \exp(-\sum_{i=1}^{N}\int_0^{t}\xi_i(r)dr)K(f_1(t,s_1),\ldots,f_N(t,s_N),\theta(t))$$

for $t \in [0, \min_{i=1,\ldots,N} s_i]$.

Ito's formula + comparing finite total variation part

for any $s_1, \ldots, s_N > t$

$$-(\sum_{i=1}^{N} \xi_i(t)) K(f_1(t, s_1), \ldots, f_N(t, s_N), \theta(t))$$

$$+\sum_{i=1}^{N} \xi_i(t) f_i(t, s_i) \frac{\partial K}{\partial x_i}(f_1(t, s_1), \ldots, f_N(t, s_N), \theta(t))$$

$$+\sum_{j=1}^{M} \eta_0^j(t) \frac{\partial K}{\partial \theta^j}(f_1(t, s_1), \ldots, f_N(t, s_N), \theta(t))$$

$$+\frac{1}{2} \sum_{i,i'=1}^{N} \sum_{k=1}^{d} \tilde{\sigma}_{i,k}(t, s_i) \tilde{\sigma}_{i',k}(t, s_{i'}) \frac{\partial^2 K}{\partial x_i \partial x_{i'}}(f_1(t, s_1), \ldots, f_N(t, s_N), \theta(t))$$

$$+\frac{1}{2}\sum_{j,j'=1}^{M}\sum_{k=1}^{d}\eta_k^j(t)\eta_k^{j'}(t)\frac{\partial^2 K}{\partial\theta_j\theta_{j'}}(f_1(t,s_1),\dots,f_N(t,s_N),\theta(t))$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{d}\tilde{\sigma}_{i,k}(t,s_i)\eta_k^j(t)\frac{\partial^2 K}{\partial x_i\theta_j}(f_1(t,s_1),\dots,f_N(t,s_N),\theta(t))$$

$$=0$$

Substitute $s_i = T_i(t, x_i)$, $i = 1, \ldots, N$

$$-(\sum_{i=1}^{N} \xi_i(t)) K(x_1, \ldots, x_N, \theta(t))$$

$$+\sum_{i=1}^{N} \xi_i(t) x_i \frac{\partial K}{\partial x_i}(x_1, \ldots, x_N, \theta(t))$$

$$+\sum_{j=1}^{M} b^j(t) \frac{\partial K}{\partial \theta^j}(x_1, \ldots, x_N, \theta(t))$$

$$+\frac{1}{2}\sum_{i,i'=1}^{N} \sum_{k=1}^{d} \sigma_{i,k}(t, x_i)\sigma_{i',k}(t, x_{i'}) \frac{\partial^2 K}{\partial x_i \partial x_{i'}}(x_1, \ldots, x_N, \theta(t))$$

$$+\frac{1}{2}\sum_{j,j'=1}^{M}\sum_{k=1}^{d}\eta_k^j(t)\eta_k^{j'}(t)\frac{\partial^2 K}{\partial\theta_j\theta_{j'}}(x_1,\ldots,x_N,\theta(t))$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{d}\sigma_{i,k}(t,x_i)\eta_k^j(t)\frac{\partial^2 K}{\partial x_i\theta_j}(x_1,\ldots,x_N,\theta(t))\} = 0$$

for any $x_1,\ldots,x_N \in (0,1)$.

Notation

$J_2 = \{(j, j') \in \{0, 1, \ldots, M\} \times \{1, \ldots, M\};\ j \leqq j'\}$

$\mathcal{C}_2$: the set of continuous functions $a : [0, 1] \times [0, 1] \to \mathbf{R}$

with $a(0, x) = a(1, x) = a(1, x) = a(x, 1) = 0$, $x \in [0, 1]$

$\mathcal{C}_1$: the set of continuous functions $\tilde{a} : [0, 1] \to \mathbf{R}$

with $a(0) = a(1) = 0$

Linear operators from $C^2((0, 1)^N \times \Theta)$ to $C((0, 1)^N \times \Theta)$

$S^{(2)}_{ii'}$, $i, i' = 1, \ldots, N$, $i < i'$,

$S^{(1)}_{ij}$, $i = 1, \ldots, N$, $j = 0, 1, \ldots, M$,

$S^{(0)}_{jj'}$, $(j, j') \in J_2$,

$$(S^{(2)}_{ii'}F)(x,\theta) = \frac{\partial^2 F}{\partial x_i \partial x_{i'}}(x,\theta), \qquad 1 \leqq i < i' \leqq N,$$

$$(S^{(1)}_{i0}F)(x,\theta) = \frac{\partial^2 F}{(\partial x_i)^2}(x,\theta), \qquad i = 1,\dots,N,$$

$$(S^{(1)}_{ij}F)(x,\theta) = \frac{\partial^2 F}{\partial x_i \partial \theta_j}(x,\theta), \qquad i = 1,\dots,N,\ j = 1,\dots,M,$$

$$(S^{(0)}_{jj'}F)(x,\theta) = \frac{\partial^2 F}{\partial \theta_j \partial \theta_{j'}}(x,\theta), \qquad 1 \leqq j \leqq j' \leqq N,$$

$$(S^{(0)}_{0j'}F)(x,\theta) = \frac{\partial F}{\partial \theta_{j'}}(x,\theta), \qquad 1 \leqq j' \leqq N,$$

for any $F \in C^2((0,1)^N \times \Theta)$.

Lemma 3 Let $N \geqq 2$, $M \geqq 1$, Θ be an open set in $\mathbf{R}^M$, and $K \in C([0,1]^N \times \Theta; [0,1])$.

Assume that K is an admissible family of copula functions and that $K|_{(0,1)^N \times \Theta}$ is C^2.

Then there is a subset A of Θ such that

the closure of A contains non-void open set in Θ, and

for any $\theta \in A$, there are $\xi_1 > 0$, $\xi_i \geqq 0$, $i = 2, \dots, N$, $\tilde{a}^{(2)}_{ii'} \in \mathcal{C}_2$, $1 \leqq i < i' \leqq N$, $\tilde{a}^{(1)}_{ij} \in \mathcal{C}_1$, $i = 1, \dots, N$, $j = 0, 1, \dots, M$, and $\tilde{a}^{(0)}_{jj'} \in \mathbf{R}$, $(j, j') \in J_2$, such that

$$\sum_{i=1}^{N} \xi_i \left(x_i \frac{\partial K}{\partial x_i}(x,\theta) - K(x,\theta)\right)$$

$$+ \sum_{1 \leqq i < i' \leqq N} \tilde{a}_{ii'}^{(2)}(x_i, x_{i'})(S_{ii'}^{(2)} K)(x,\theta) + \sum_{i=1}^{N}\sum_{j=0}^{M} \tilde{a}_{ij}^{(1)}(x_i)(S_{ij}^{(1)} K)(x,\theta)$$

$$+ \sum_{(j,j') \in J_2} \tilde{a}_{jj'}^{(0)}(S_{jj'}^{(0)} K)(x,\theta) = 0, \text{ for all } x = (x_1, \ldots, x_N) \in (0,1)^N$$

and

$$\sum_{1 \leqq i < i' \leqq N} \tilde{a}_{ii'}^{(2)}(x_i, x_{i'}) z_i z_{i'} + \sum_{i=1}^{N} \tilde{a}_{i0}^{(1)}(x_i) z_i^2 + \sum_{i=1}^{N}\sum_{j=1}^{M} \tilde{a}_{ij}^{(1)}(x_i) z_i y_j$$

$$+ \sum_{(j,j') \in J_2} \tilde{a}_{jj'}^{(0)} y_j y_{j'} \geqq 0 \ \forall x \in (0,1)^N, \ \forall z_1, \ldots, z_N, y_1, \ldots, y_M \in \mathbf{R}$$

How to verify that the condition in Lemma does not hold

$n \geqq 1$

$\vec{z} = (z_{ik})_{i=1,\dots,N,k=1,\dots,n} \in (0,1)^{nN}$

For $\vec{k} = (k_1, \dots, k_N) \in \{1, \dots, n\}^N$, and $\vec{z} \in (0,1)^{nN}$,

$Z_i(\vec{z}, \vec{k}) = z_{ik_i}$, $i = 1, \dots, N$,

$\vec{Z}(\vec{z}, \vec{k}) = (z_{1k_1}, \dots, z_{Nk_N}) \in (0,1)^N$.

$$\sum_{i=1}^{N} \xi_i (z_{ik_i} \frac{\partial K}{\partial x_i}(\vec{Z}(\vec{z},\vec{k}),\theta) - K(\vec{Z}(\vec{z},\vec{k}),\underline{\theta}))$$

$$+ \sum_{1 \leqq i < i' \leqq N} \tilde{a}_{ii'}^{(2)}(z_{ik_i}, z_{i'k_{i'}})(S_{ii'}^{(2)} K)(\vec{Z}(\vec{z},\vec{k}),\underline{\theta})$$

$$+ \sum_{i=1}^{N} \sum_{j=0}^{M} \tilde{a}_{ij}^{(1)}(z_{ik_i})(S_{ij}^{(1)} K)(\vec{Z}(\vec{z},\vec{k}),\underline{\theta}) + \sum_{(j,j') \in J_2} \tilde{a}_{jj'}^{(0)}(S_{jj'}^{(0)} K)(\vec{Z}(\vec{z},\vec{k}),\underline{\theta}) = 0$$

for $\vec{k} = (k_1, \ldots, k_N) \in \{1, \ldots, n\}^N$

The number of equations $= n^N$

Unknown factors : ξ_i, $\tilde{a}_{ii'}^{(2)}(z_{ik_i}, z_{i'k_{i'}})$, $\tilde{a}_{ij}^{(1)}(z_{ik_i})$, $\tilde{a}_{jj'}^{(0)}$

Number of Unknown factors $= O(n^2)$

$$C_n^{(2)} = \{(i,i') \in \{1,2,\ldots,N\}^2;\ i < i'\} \times \{1,2,\ldots,n\}^2$$

$$C_n^{(1)} = \{1,2,\ldots,N\} \times \{0,1,\ldots,M\} \times \{1,2,\ldots,n\}.$$

For any $G \in C^2((0,1)^N \times \Theta)$, $n \geqq 1$, and $\vec{k} \in \{1,\ldots,n\}^N$,
we define continuous functions defined in $(0,1)^{nN} \times \Theta$,
$(M_i^{(n)I} G)(\cdot,\vec{k})$, $i = 1,\ldots,N$,
$(M_{ii'pq}^{(n)(2)} G)(\cdot,\vec{k})$, $(i,i',p,q) \in C_n^{(2)}$,
$(M_{ijp}^{(n)(1)} G)(\cdot,\vec{k})$, $(i,j,p) \in C_n^{(1)}$,
$(M_{jj'}^{(n)(0)} G)(\cdot,\vec{k})$, $(j,j') \in J_2$, by

$$(M_i^{(n)I} G)(\vec{z},\theta,\vec{k}) = Z_i(\vec{z},\vec{k}) \frac{\partial G}{\partial x_i}(\vec{Z}(\vec{z},\vec{k}),\theta) - G(\vec{Z}(\vec{z},\vec{k}),\theta) \quad i = 1,\ldots,N,$$

$$(M_{ii'pq}^{(n)(2)} G)(\vec{z},\theta,\vec{k}) = \delta_{p,k_i}\delta_{q,k_{i'}}(S_{ii'}^{(2)} G)(\vec{Z}(\vec{z},\vec{k}),\theta) \quad (i,i',p,q) \in C_n^{(2)},$$

$$(M^{(n)(1)}_{ijp}G)(\vec{z},\theta,\vec{k}) = \delta_{p,k_i}(S^{(1)}_{ij}G)(\vec{Z}(\vec{z},\vec{k}),\theta), \quad (i,j,p) \in C^{(1)}_n,$$

$$(M^{(n)(0)}_{jj'}G)(\vec{z},\theta,\vec{k}) = (S^{(0)}_{jj'}G)(\vec{Z}(\vec{z},\vec{k}),\theta), \quad (j,j') \in J_2,$$

for any $\vec{z} \in (0,1)^{nN}$ and $\theta \in \Theta$.

$$C_{n0} = C_n^{(2)} \cup C_n^{(1)} \cup J_2$$
$$C_n = \{1, \ldots, N\} \cup C_{n0}.$$

$$\#(C_{n0}) = n^2 N(N-1)/2 + nN(M+1) + M(M+3)/2$$

$$\#(C_n) = N + \#(C_{n0}).$$

For any $G \in C^2((0,1)^N \times \Theta)$, $n \geqq 1$, and $\gamma \in C_n$ we define a continuous function $(\vec{M}^{(n)}G)_\gamma : (0,1)^{nN} \times \Theta \to \mathbf{R}^{\{1,\ldots,n\}^N}$ by

$$(\vec{M}^{(n)}G)_\gamma(\vec{z}, \theta)$$

$$= \begin{cases} ((M_{ii'pq}^{(n)(2)}G)(\vec{z}, \theta, \vec{k}))_{\vec{k}\in\{1,\ldots,n\}^N} & \text{if } \gamma = (i, i', p, q) \in C_n^{(2)}, \\ (M_{ijp}^{(n)(1)}G)(\vec{z}, \theta, \vec{k}))_{\vec{k}\in\{1,\ldots,n\}^N} & \text{if } \gamma = (i, j, p) \in C_n^{(1)}, \\ (M_{jj'}^{(n)(0)}G)(\vec{z}, \theta, \vec{k}))_{\vec{k}\in\{1,\ldots,n\}^N} & \text{if } \gamma = (j, j') \in J_2, \\ ((M_i^{(n)I}G))(\vec{z}, \theta, \vec{k}))_{\vec{k}\in\{1,\ldots,n\}^N} & \text{if } \gamma = i \in \{1, \ldots, N\}. \end{cases}$$

For any $G \in C^2((0,1)^N \times \Theta)$, $n \geqq 1$, $\vec{z} \in (0,1)^{nN}$ and $\theta \in \Theta$,

let $V_n(G, \vec{z}, \theta)$ (resp. $V_{n0}(G, \vec{z}, \theta)$) be the vector subspace of $\mathbf{R}^{\{1,\ldots,n\}^N}$

spaned by $\{(\vec{M}^{(n)}G)_\gamma(\vec{z}, \theta);\ \gamma \in C_n\}$

(resp. $\{(\vec{M}^{(n)}G)_\gamma(\vec{z}, \theta);\ \gamma \in C_{n0}\}$).

Also, let $N_{(n)}(G, \vec{z}, \theta)$ be a vector space in $\mathbf{R}^N$ given by

$$N_{(n)}(G, \vec{z}, \theta) = \{(v_1, \ldots, v_N) \in \mathbf{R}^N;\ \sum_{i=1}^N v_i (\vec{M}^{(n)}G)_i(\vec{z}, \theta) \in V_{n0}(G, \vec{z}, \theta)\}.$$

Lemma 4 Let $N \geqq 2$, $M \geqq 1$, and Θ be an open subset of $\mathbf{R}^M$ Let $K \in C([0,1]^N \times \Theta; [0,1])$. Assume that K is an admissible family of copula functions, and that $K|_{(0,1)^N \times \Theta}$ is C^2. Then there is a subset A of Θ such that the closure of A contains non-void open set in Θ, and for any $\theta \in A$ and $\vec{z} \in (0,1)^{nN}$, $N_{(n)}(K, \vec{z}, \theta) \cap [0, \infty)^N \neq \{0\}$.

As a corollary we have the following.

Corollary 5 Let $N \geqq 2$, $M \geqq 1$, and Θ be an open subset of $\mathbf{R}^M$. Let $K \in C([0,1]^N \times \Theta; [0,1])$. Assume that K is an admissible family of copula functions, and that $K|_{(0,1)^N \times \Theta}$ is C^2. Then for any $n \geqq 1$ and $\vec{z} \in (0,1)^{nN}$, there is a non-void open subset U of Θ such that

$$\dim V_n(K, \vec{z}, \theta) \leqq \#(C_n) - 1, \qquad \theta \in U.$$

Remark 6 Let $N \geqq 3$, $M \geqq 1$, and Θ be an open subset of $\mathbf{R}^M$. Let $K \in C([0,1]^N \times \Theta; [0,1])$. Assume that K is an admissible family of copula functions, and that $K|_{(0,1)^N \times \Theta}$ is C^2.

Assume moreover that

Θ is connected and that

$K(x, \cdot) : \Theta \to \mathbf{R}$ is real analytic.

If there exist $n \geqq 1$, $\vec{z} \in (0,1)^{nN}$, and $\theta_0 \in \Theta$ such that

$$\dim V_n(K, \vec{z}, \theta_0) = \#(C_n),$$

then K is not admissible family of copula functions.

Special case:

K is a family of Archimedean copula functions, i.e.,

$\exists\, \varphi : (0,1)\times\Theta \to (0,\infty)\ \exists \rho : (0,\infty)\times\Theta \to (0,1)$

$$K(x_1,\ldots,x_N,\theta) = \rho(\sum_{k=1}^{N}\varphi(x_k,\theta),\theta), \qquad x_1,\ldots,x_N \in (0,1),\ \theta\in\Theta.$$

$\rho(\cdot,\theta)$ = the inverse function of $\varphi(\cdot,\theta)$

Remark 7 Let

$$m_0 = \frac{N(N-1)}{2}n^2 + N(M+3-N)n - \frac{(N-1)(2M+4-N)}{2} + \frac{M(M+3)}{2}.$$

Assume that Θ is connected and that $\varphi : (0,1)\times\Theta \to (0,\infty)$ is real analytic. If there exists a $\theta_0 \in \Theta$ such that $\dim V_n(K,\vec{z},\theta_0) = m_0 + 1$, then K is not an admissible family of copula functions.

a family of Gumbel copula functions

$N = 3$, $M = 1$, $\Theta = (0, 1)$.

$\varphi(x, \theta) = (-\log x)^{\theta}$, $\rho(y, \theta) = \exp(-y^{1/\theta})$,

$n = 5 \;\; m_0 = 89$.

$\dim V_n(K, \vec{z}, \theta_0) = 90$ for $(z_{i1}, \ldots, z_{i5}) = (0.55, 0.65, 0.75, 0.85, 0.95)$,
$i = 1, 2, 3$, and $\theta_0 = 0.4$ or 0.6.

by applying Householder transformation